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Derivation of the Fan Laws

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ABSTRACT

The purpose of this white paper is to provide a derivation of the fan laws and to briefly discuss their application.

Fan engineering is macroscopic. It is based on the assumption that a large number of particles are distributed continuously throughout space. Therefore, the working matter is not viewed as separate particles of oxygen, nitrogen, water, etc., but as a continuous fluid. From this standpoint, it is possible to discuss the state of the fluid at a particular point in space.

The "state" of anything always is described quantitatively. According to System Internationale (SI), there are seven basic or primary physical quantities or dimensions. They are:

- Mass (M)
- Length (L)
- Time (T)
- Thermodynamic temperature
- Intensity of electric current
- Luminous intensity
- Amount of substance

All other quantities are secondary and can be derived from the seven primary dimensions. Some secondary physical quantities and their dimensions that are important in fluid dynamics are:

- Surface, L^2
- Volume, L^3
- Density, M/L^3
- Velocity, L/T
- Volumetric flow rate, L^3/T
- Acceleration, L/T^2
- Momentum, ML/T
- Force, ML/T^2
- Pressure, M/LT^2
- Energy, ML/T^2
- Pressure head, L^2/T^2
- Power, ML^2/T^3
- Moment of inertia, ML^2
- Angular velocity, $1/T$
- Angular acceleration, $1/T^2$
- Angular momentum, ML^2/T
- Torque, ML^2/T^2
- Modulus of elasticity, M/LT^2
- Surface tension, M/T^2
- Viscosity (absolute), M/LT
- Viscosity (kinematic), L^2/T

Experience shows that eight of these physical quantities are significant in describing the operating state of a fan or, more generally, the operating state of a turbomachine. The quantities are:

- Q, volumetric rate of flow through the fan
- h, head developed by the fan
- H, power
- N, fan speed (angular velocity of the rotor)
- D, rotor diameter (or any characteristic dimension)
- ρ, density
- μ, viscosity
- Y, elasticity

For every operating state of a fan there exists a quantitative combination of these variables to describe that state. The combinations of these variables are not random. A relationship that has yet to be determined exists between all of these variables. For example, we may say the flow rate, Q, is a function of the other variables, or:

$$Q = f(h, H, N, D, \rho, \mu, Y)$$

To determine this relationship experimentally, one would have to vary each of the quantities while holding the others steady and measure Q to determine the effect of the variable on Q. This process would be very long and unmanageable.

Fortunately, there exists a procedure in fluid mechanics called dimensional analysis that formulates a relationship between the variables and allows prediction of the operating state. The procedure basically consists of combining the variables into groups called PI groups. The groups are unrelated to 3.1416 and have no dimensions.

Buckingham's PI theorem states there can be only a certain number of PI groups. That number is equal to n minus k, where n is equal to the number of variables describing the function and k is equal to the number of primary dimensions used by the original variables. In our case, the number of variables describing the function is eight, and all of the variables can be described by the dimensions M, L, and T. Hence, k is equal to three. Therefore, there are five PI groups.

Calculation of the PI groups for a turbomachine is as follows:

First, select three variables that, when combined, will not form a dimensionless group. Let's first select N, D, and ρ. These three variables taken together contain at least one of each of the primary dimensions found in the eight original variables and, when combined, are not dimensionless.

$$(N)(D)(\rho) = (1/T)(L)(M/L^3) = M/TL^2$$

Next, raise each of these variables to an as-yet-unknown power and combine them with one of the five remaining variables raised to the first power.

$$PI_1 = (N)^a(D)^b(\rho)^c(Q)^1 = (1/T)^a(L)^b(M/L^3)^c(L^3/T)^1 = M^0L^0T^0 = 1$$

Simultaneously solve the exponents to create a dimensionless product:

M:	$c = 0$	$c = 0$
L:	$b - 3c + 3 = 0$	$b = -3$
T:	$-a - 1 = 0$	$a = -1$

Therefore:

$$PI_1 = Q/ND^3 \qquad \text{flow coefficient}$$

Repeating:

$$PI_2 = (N)^a(D)^b(\rho)^c(h)^1 = (1/T)^a(L)^b(M/L^3)^c(L^2/T^2)^1 = M^0L^0T^0$$

M:	$c = 0$	$c = 0$
L:	$b - 3c + 2 = 0$	$b = -2$
T:	$-a - 2 = 0$	$a = -2$

$$PI_2 = h/N^2D^2$$

Head is defined as energy per unit mass. A quantity more easily measured is pressure. By manipulating dimensions, it can be shown that head actually is pressure divided by density.

$$h = L^2/T^2 = ML^2/T^2 \times 1/M = M/LT^2 \times L^3/M = P/\rho$$

(energy/mass) (pressure/density)

Therefore, a more convenient form of PI_2 is:

$PI_2 = P/\rho N^2D^2$ pressure coefficient

$PI_3 = (N)^a(D)^b(\rho)^c(H)^1 = (1/T)^a(L)^b(M/L^3)^c(ML^2/T^3)^1 = M^0L^0T^0$

M:	c + 1 = 0	c = -1
L:	b – 3c + 2 = 0	b = -5
T:	-a – 3 = 0	a = -3

Solving to get:

$PI_3 = H/N^3 D^5 \rho$ power coefficient

$PI_4 = (N)^a(D)^b(\rho)^c(\mu)^1 = (1/T)^a(L)^b(M/L^3)^c(M/LT)^1 = M^0L^0T^0$

M:	c + 1 = 0	c = -1
L:	b – 3c – 1 = 0	b = -2
T:	-a – 1 = 0	a = -1

Solving to get:

$PI_4 = \mu/ND^2\rho$ Add π to get fan Reynolds number

$PI_5 = (N)^a(D)^b(\rho)^c(Y)^1 = (1/T)^a(L)^b(M/L^3)^c(M/LT^2)^1 = M^0L^0T^0$

M:	c + 1 = 0	c = -1
L:	b – 3c – 1 = 0	b = -2
T:	-a – 2 = 0	a = -2

And finally:

$PI_5 = Y/N^2D^2\rho$

Because PI numbers are dimensionless, they can be combined to form new PIs (which also are dimensionless), such as specific speed:

$PI_6 = (PI_1)^{.5}/(PI_2)^{.75} = (NQ^{.5}\rho^{.75})/(P^{.75})$

… and specific diameter:

$PI_7 = (PI_2)^{.25}/(PI_1)^{.5} = (DP^{.25})/(\rho^{.25}Q^{.5})$

Specific speed and specific diameter are important because, for every operating condition for a given fan design, there is only one specific speed and one specific diameter. That means there is only one specific speed and one specific diameter for a given fan design's peak efficiency. This concept is useful when the operating condition is known but the fan design has not yet been chosen because, after the specific speed or specific diameter has been calculated, the fan design with the highest efficiency at that specific speed or diameter is obvious.

The first three PI groups lead directly to three important fan laws:

$$Q_2 = Q_1(N_2/N_1)^1(D_2/D_1)^3$$

$$P_2 = P_1(\rho_2/\rho_1)^1(N_2/N_1)^2(D_2/D_1)^2$$

$$H_2 = H_1(\rho_2/\rho_1)^1(N_2/N_1)^3(D_2/D_1)^5$$

The fan laws provide a direct relationship between operating states of a fan. It is this relationship that allows us to predict fan performance from only one tested fan curve (instead of many) and one model, provided three conditions of similarity exist between model and prototype. The conditions are:

- Geometric similarity—Fluid flow between model and prototype must have geometrically similar boundaries as well as angular equality.
- Kinematic similarity—Streamlines of the fluid are similar between model and prototype.
- Dynamic similarity—The force distribution between flows must be parallel and have the same ratio at all points in the flow.

The fan laws give us a relatively simple method of predicting fan performance by relating the performance of a model to that of a prototype without complex mathematical equations and with a reasonably limited amount of testing.

For example, what would happen if the capacity of a fan in an air system ($P = kQ^2$) were 24,000 cfm and we increased its speed from 1180 rpm to 1750 rpm? Assume a 24½-in. impeller diameter.

$$Q_2 = 24{,}000 \text{ cfm } (1750/1180)^1(24½/24½)^3 = 35{,}593 \text{ cfm}$$

Easy. While the derivation of the fan laws may seem abstract, the final product is simple and provides a powerful tool in fan application.

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